Example 1:

$$f(x) = \frac{x^3}{(2-x)^2}$$

Start with $\displaystyle \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$.

$$\frac{1}{1-x} = (1-x)^{-1} = \sum_{n=0}^{\infty} x^n$$

Differentiate

$$\frac{d}{dx}(1-x)^{-1} = \frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right)$$

$$= (1-x)^{-2} = \sum_{n=0}^{\infty} n x^{n-1}$$

$$(1-x)^{-2} = \sum_{n=0}^{\infty} n x^{n-1}$$

$$\|$$

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} n x^{n-1}$$

We want $\dfrac{x^3}{(2-x)^2}$, so

$$\frac{1}{(2-x)^2} = \frac{1}{\left(2\left(1-\frac{x}{2}\right)\right)^2} = \frac{1}{4\left(1-\frac{x}{2}\right)^2}$$

so $$\frac{x^3}{(2-x)^2} = \frac{x^3}{4} \cdot \frac{1}{\left(1-\frac{x}{2}\right)^2}$$

$$\boxed{\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} n x^{n-1}}$$

$$\frac{x^3}{(2-x)^2} = \frac{x^3}{4} \cdot \frac{1}{\left(1-\frac{x}{2}\right)^2}$$

$$\frac{1}{\left(1-\frac{x}{2}\right)^2} = \boxed{\sum_{n=0}^{\infty} n\left(\frac{x}{2}\right)^{n-1}}$$

Multiply by $\frac{x^3}{4}$ and we are done,

$$\boxed{\frac{x^3}{(2-x)^2} = \frac{x^3}{4} \cdot \frac{1}{\left(1-\frac{x}{2}\right)^2} = \frac{x^3}{4} \cdot \sum_{n=0}^{\infty} n\left(\frac{x}{2}\right)^{n-1}}$$

$$\frac{x^3}{(2-x)^2} = \frac{x^3}{4}\sum_{n=0}^{\infty} n\left(\frac{x}{2}\right)^{n-1}$$

Combine powers, determine coefficients

$$\frac{x^3}{4}\sum_{n=0}^{\infty} n\left(\frac{x}{2}\right)^{n-1} = \sum_{n=0}^{\infty} \frac{x^3}{4}\cdot\left(\frac{x}{2}\right)^{n-1}\cdot n$$

$$= \sum_{n=0}^{\infty} n\,\frac{x^3}{4}\cdot\frac{x^{n-1}}{2^{n-1}}$$

$$= \sum_{n=0}^{\infty} n\,\frac{x^3}{2^2}\,\frac{x^{n-1}}{2^{n-1}}$$

$$= \sum_{n=0}^{\infty} n\,\frac{x^{n+2}}{2^{n+1}}$$

$$\sum_{n=0}^{\infty} n \frac{x^{n+2}}{2^{n+1}} = \frac{x^3}{(2-x)^2}$$

$$\parallel$$

$$\left(0 \cdot \frac{x^2}{2}\right) + \left(\frac{1 \cdot x^3}{4}\right) + \left(\frac{2 \cdot x^4}{8}\right) + \left(\frac{3 \cdot x^5}{16}\right) + \cdots$$

$$+ 0 + 0 \cdot x$$

$$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

$$a_0 = 0 = a_1 = a_2$$

$$a_3 = \frac{1}{4}, \quad a_4 = \frac{1}{4}, \quad a_5 = \frac{3}{16}$$

One More:

$$\sum_{n=0}^{\infty} \frac{x^{3n+1}}{2(n!)^3}$$

$$= 0 + \frac{x}{2} + 0 \cdot x^2 + 0 \cdot x^3 + \frac{x^4}{2}$$

$$+ 0 \cdot x^5 + 0 \cdot x^6 + \frac{x^7}{16} + \text{—}$$

$$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \text{—}$$

$a_0 = 0 \quad a_1 = \frac{1}{2} \quad a_2 = 0 \quad a_3 =$

$a_4 = \frac{1}{2} \quad a_5 = 0 \quad a_6 = 0 \quad a_7 = \frac{1}{16}$

Taylor Series

When does our series equal a

function we know?

$$y'(t) = y(t)$$

$$y(t) = e^t$$

or if we write with x's

$\boxed{y(x) = e^x}$ is a solution,

Try $y(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

We showed in class this series has radius of convergence $= \infty$.

$$y'(x) = y(x)$$

$$y'(x) = \sum_{n=0}^{\infty} \frac{d}{dx}\left(\frac{x^n}{n!}\right)$$

$$= \sum_{n=1}^{\infty} \frac{n\, x^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{\cancel{n}\, x^{n-1}}{\cancel{n} \cdot (n-1)!}.$$

$$= \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!}$$

$m = n-1$

$n = m+1$

$$y'(x) = \sum_{m=0}^{\infty} \frac{x^m}{m!}.$$

$$= y(x)$$

We now suspect this function $\sum_{m=0}^{\infty} \frac{x^m}{m!} = e^x$.

This is true!

Tangent Lines from calc 1

If $y = f(x)$ and

y is differentiable at $x = a$,

then the tangent line

to $y = f(x)$ at $x = a$ is

$$y - f(a) = f'(a) \cdot (x - a)$$

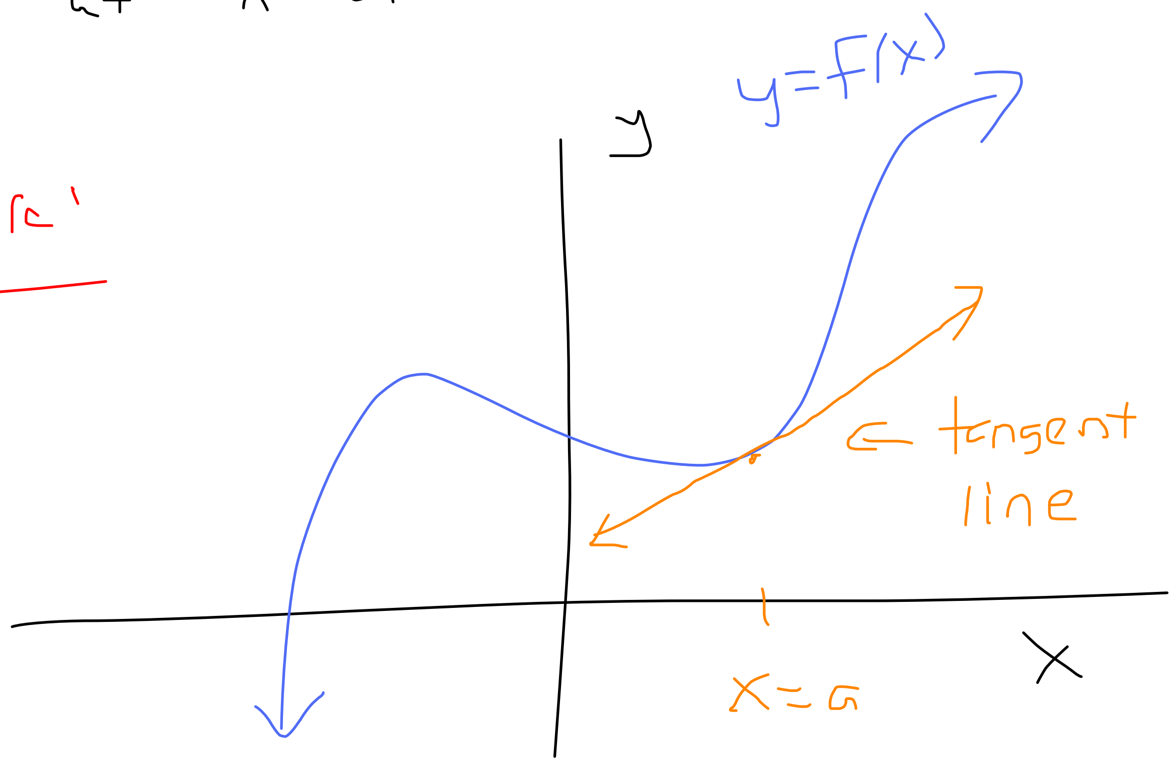
$$y = f(a) + f'(a) \cdot (x - a)$$

When $x = a$ $\quad y = f(a)$

At $x=a$, the tangent line equals the original function

It is the best linear approximation to $y=f(x)$ at $x=a$

Picture:



Quadratic Approximation

The best quadratic approximation to $y = f(x)$ at $x = a$ is

tangent line

$$y = \frac{f''(a)}{2}(x-a)^2 + \boxed{f'(a)(x-a) + f(a)}$$

when $x = a$, $y = f(a)$

$$y' = f''(a)(x-a) + f'(a)$$

when $x = a$

$$y' = f'(a)$$

Cubic Approximation?

There is one and it is

(to $y=f(x)$ at $x=a$)

$$y = \underbrace{\boxed{f(a)+f'(a)(x-a)+\frac{f''(a)}{2}(x-a)^2}}_{\text{quadratic approximation}} + \frac{f'''(a)}{6\,(=3!)}(x-a)^3$$

$$y' = f'(a) + f''(a)(x-a) + \frac{f'''(a)}{2}(x-a)^2$$

$$y'(a) = f'(a)$$

$$y'' = f''(a) + f'''(a)(x-a)$$

$$y''(a) = f''(a)$$ even better

Taylor Series

f infinitely differentiable on

$(c-R, c+R)$ ($R>0$, could be ∞),

then on $(c-R, c+R)$,

$$\boxed{f(x) = \sum_{n=0}^{\infty} f^{(n)}(c) \frac{(x-c)^n}{n!}}$$

Taylor Series centered at c.

where $f^{(n)} = n^{th}$ derivative of f

$f^{(0)} = f$

MacLaurin Series: Taylor series when $c = 0$,

$$f(x) = \sum_{n=0}^{\infty} f^{(n)}(0) \frac{x^n}{n!}$$

Example 2: (exponential)

$$f(x) = e^x$$

Find the MacLaurin series

$$\sum_{n=0}^{\infty} \boxed{f^{(n)}(0)} \cdot \frac{x^n}{n!}$$

$\rightarrow$ figure this out!

Once we know $f^{(n)}(0)$, we'll

know the series.

$$f^{(0)}(0) = f(0) = e^0 = 1$$

$$f'(x) = e^x, \quad f'(0) = 1$$

$$f''(x) = e^x, \quad f''(0) = 1$$

$$f'''(x) = e^x, \quad f'''(0) = 1$$

$$\boxed{f^{(n)}(x) = e^x, \quad f^{(n)}(0) = 1}$$

replace $f^{(n)}(0)$ with 1 in the equation for the series

Don't do anything else!

$$\boxed{e^x = \sum_{n=0}^{\infty} 1 \cdot \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{n!}}$$

for all real numbers x

Know this series!

Example 3: (sinc)

Find MacLaurin series for $f(x) = \sin(x)$,

Find $f^{(n)}(0)$!

Plug into formula

$$\sum_{n=0}^{\infty} f^{(n)}(0) \frac{x^n}{n!}$$

$$f(x) = \sin(x) = f^{(0)}(x), \quad f(0) = 0$$

$$f'(x) = \cos(x), \quad f'(0) = 1$$

$$f''(x) = -\sin(x), \quad f''(0) = 0$$

$$f'''(x) = -\cos(x), \quad f'''(0) = -1$$

$$f^{(4)}(x) = \sin(x)$$ back to where we started

$$f^{(4)}(0) = 0$$

$$f^{(n)}(0) = \begin{cases} 0, & n \text{ even} \\ \pm 1, & n \text{ odd} \end{cases}$$

How to tell the sign when n is odd?

To find this out:

$$\sin(x) = \sum_{n=0}^{\infty} f^{(n)}(0)\,\frac{x^n}{n!}$$

but all even exponents disappear

Every odd number is equal to $2k+1$ for some counting number k ($k=0$ included),

$$\sin(x) = \sum_{k=0}^{\infty} f^{(2k+1)}(0)\,\frac{x^{2k+1}}{(2k+1)!}$$

removed all even exponents

Back to the pattern,

$$f'(0) = 1$$

$$f'''(0) = -1$$

$$f^{(5)}(0) = 1$$

$$f^{(7)}(0) = -1$$

$$(-1)^k = f^{(2k+1)}(0)$$

$$\boxed{\sin(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}}$$

Know this series!

We want the series for $\cos(x)$ — but I don't want to work.

Fact! If $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$

on $(c-R, c+R)$ where $R > 0$

and f is infinitely differentiable on $(c-R, c+R)$, then

$$\boxed{a_n = \frac{f^{(n)}(c)}{n!}}$$

Example 4. (cosine)

Maclaurin Series for $f(x) = \cos(x)$

We know $\sin(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$.

Take derivative of both sides $= \frac{d}{dx}\left(x^{2k+1}\right)$

$$\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \boxed{x^{2k} \cdot (2k+1)}}{(2k+1)!}$$

$$= \boxed{\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}}$$

Know this series

Taylor Polynomials

Includes tangent line, quadratic & cubic approximations.

Given $y = f(x)$, differentiable k times on $(c-R, c+R)$ $(R>0)$,

the k^{th} Taylor polynomial about c

is $$T_{k,c}(x) = \sum_{n=0}^{k} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

(cut off Taylor series at k)

Let $R_{k,c}(x) = f(x) - T_{k,c}(x)$.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{n}(c)}{n!}(x-c)^{n}$$

on $(c-R, c+R)$ if

$\lim_{k \to \infty} R_{k,c}(x) = 0$ for

all x in $(c-R, c+R)$

Example 5: Find the sum:

$$\sum_{n=1}^{\infty} \frac{\pi^{2n+1}}{(2n)!\, 36^{n}}$$